

Hopf surfaces in locally conformally Kähler manifolds with potential

Liviu Ornea¹, and Misha Verbitsky²

Abstract

An LCK manifold with potential is a compact quotient M of a Kähler manifold X equipped with a positive plurisubharmonic function f , such that the monodromy group acts on X by holomorphic homotheties and maps f to a function proportional to f . It is known that M admits an LCK potential if and only if it can be holomorphically embedded to a Hopf manifold. We prove that any non-Vaisman LCK manifold with potential contains a Hopf surface H . Moreover, H can be chosen non-diagonal, hence, also not admitting a Vaisman structure.

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1 Introduction: LCK manifolds

Let (M, I) be a complex manifold, $\dim_{\mathbb{C}} M \geq 2$. It is called **locally conformally Kähler** (LCK) if it admits a Hermitian metric g whose fundamental 2-form $\omega(\cdot, \cdot) := g(\cdot, I\cdot)$ satisfies

$$d\omega = \theta \wedge \omega, \quad d\theta = 0, \quad (1.1)$$

for a certain closed 1-form θ called **the Lee form**.

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Definition (1.1) is equivalent to the existence of a covering \tilde{M} endowed with a Kähler metric Ω which is acted on by the deck group $\text{Aut}_M(\tilde{M})$ by homotheties. Let

$$\chi : \text{Aut}_M(\tilde{M}) \longrightarrow \mathbb{R}^{>0}, \quad \chi(\tau) = \frac{\tau^* \Omega}{\Omega}, \quad (1.2)$$

be the group homomorphism which associates to a homothety its scale factor.

For definitions and examples, see [DO] and our more recent papers.

An LCK manifold (M, ω, θ) is called **Vaisman** if $\nabla \theta = 0$, where ∇ is the Levi-Civita connection of g . The main example of Vaisman manifold is the diagonal Hopf manifold ([OV3]). The Vaisman compact complex surfaces are classified in [Be].

Note that there exist compact LCK manifolds which do not admit Vaisman metrics. Such are the LCK Inoue surfaces, [Be], the Oeljeklaus-Toma manifolds, [Kas], and the non-diagonal Hopf manifolds, [OV3].

It is known that on any Vaisman manifold, the following formula holds, [Va2], [DO]:

$$d^c \theta = \theta \wedge \theta^c - \omega. \quad (1.3)$$

Moreover, one can see, [Ve], that the $(1,1)$ -form $\omega_0 := -d^c \theta$ is semi-positive definite, having all eigenvalues¹ positive, but one which is 0.

An LCK manifold is called **with potential** if it admits a Kähler covering on which the Kähler metric has a global and positive potential function which is acted on by holomorphic homotheties by the deck group. Among the examples: all Vaisman manifolds, but also non-Vaisman ones, such as the non-diagonal Hopf manifolds, [OV1], [OV4]. On the other hand, there exist compact LCK manifolds which cannot admit LCK potential, e.g. Inoue surfaces (see [OV1]) and their higher dimensional analogues, the Oeljeklaus-Toma manifolds, see Remark 3.10.

One can prove, [OV4], that on a compact manifold, a positive, automorphic potential can always be deformed to a **proper** positive, automorphic potential. The existence of such a potential is equivalent with the image of the character χ being isomorphic with \mathbb{Z} . In this case, the LCK manifold with potential is called **of LCK rank 1**.

As on the Kähler covering of an LCK manifold with potential, one has $\pi^* \omega = \psi^{-1} d d^c \psi$, where the potential is $\psi = e^{-v}$ and the Lee form is $\pi^* \theta = d v$, one can prove ([OV2], also [AD]):

Claim 1.1: Equation (1.3) is satisfied on LCK manifolds with potential.

¹The eigenvalues of a Hermitian form η are the eigenvalues of the symmetric operator L_η defined by the equation $\eta(x, Iy) = g(Lx, y)$.

The aim of this paper is to prove that any compact non-Vaisman LCK manifold with potential contains a holomorphically embedded Hopf surface which is non-diagonal (this is the same as being non-Vaisman). As an application, we show that a compact LCK manifold with potential (M, ω, θ) is Vaisman if and only if the form $d^c \theta$ is sign semi-definite.

2 The form ω_0 on a compact LCK manifold with potential

In general, on an LCK manifold M with potential ψ on \tilde{M} , the norm (w.r.t. the LCK metric) of the Lee form $d\psi$ is not constant. The constancy of the norm of the Lee form is equivalent to the LCK metric being Gauduchon (see Proposition 2.3 below) and Vaisman, as shown in [MM].

Definition 2.1: On a complex manifold of complex dimension n , a Hermitian metric whose Hermitian 2-form ω satisfies the equation $\partial\bar{\partial}\omega^{n-1} = 0$ is called **Gauduchon**.

Remark 2.2: On a compact Hermitian manifold, a Gauduchon metric exists in each conformal class and it is unique up to homothety. Moreover, it is characterized by the co-closedness of its Lee form. A Vaisman metric is a Gauduchon metric in its conformal class, [G].

Proposition 2.3: Let (M, ω, θ) be a compact LCK manifold with potential. Then the LCK form ω is Gauduchon if and only if $|\theta| = \text{const}$.

Proof: The Hermitian form ω is Gauduchon if and only if $dd^c \omega^{n-1} = 0$.

We compute $dd^c \omega^{n-1}$ using equation (1.3) which is satisfied on an LCK manifold with potential (Claim 1.1). This gives

$$dd^c \omega^{n-1} = (n-1)^2 \omega^{n-1} \wedge \theta \wedge \theta^c + (n-1) \omega^{n-1} \wedge d\theta^c.$$

On the other hand,

$$\omega^{n-1} \wedge \theta \wedge \theta^c = \frac{1}{n} |\theta|^2 \omega^n$$

and

$$d\theta^c \wedge \omega^{n-1} = -\omega \wedge \omega^{n-1} + \theta \wedge \theta^c \wedge \omega^{n-1} = \left(\frac{1}{n} |\theta|^2 - 1 \right) \omega^n.$$

All in all we get:

$$dd^c \omega^{n-1} = \frac{(n-1)^2}{n} |\theta|^2 \omega^n + (n-1) \left(\frac{1}{n} |\theta|^2 - 1 \right) \omega^n = (n-1) (|\theta|^2 - 1) \omega^n.$$

Then $dd^c \omega^{n-1} = 0$ if and only if $|\theta| = 1$. This finishes the proof. ■

Observe now that the eigenvalues of $\omega_0 = -d^c \theta$ are 1 (with multiplicity $n-1$) and $1 - |\theta|^2$. As ω_0 is exact on a compact manifold, its top power cannot be sign-definite (Stokes theorem). Two possibilities occur:

1. $|\theta|$ is non-constant, and then $1 - |\theta|^2$ has to change sign on M ;
2. $|\theta| = \text{const.}$ and then $|\theta| = 1$ and ω_0 is semi-positive definite.

We obtained the following corollary.

Corollary 2.4: Let (M, ω, θ) be a compact LCK manifold with potential. Then the LCK metric is Gauduchon if and only if $\omega_0 = -d^c \theta$ is semi-positive definite.

Remark 2.5: Our interest in studying the form ω_0 on compact LCK manifolds arose from the attempt to clarify the relation between the pluricanonical condition $(\nabla \theta)^{1,1} = 0$, equivalent with $d^c \theta = \theta \wedge \theta^c - |\theta|^2 \omega$, introduced in [Kok], and the existence of a positive, automorphic potential. In fact, from the above it can be seen that a compact LCK manifold with potential, and with constant norm of θ is pluricanonical, and hence, by [MM], it is Vaisman.

3 Hopf surfaces in LCK manifolds with potential

3.1 Complex surfaces of Kähler rank 1

Definition 3.1: ([HL]) A compact complex surface is **of Kähler rank 1** if and only if it is not Kähler but it admits a closed semipositive $(1,1)$ -form whose zero locus is contained in a curve.

Lemma 3.2: A compact LCK surface M with potential and semi-positive form ω_0 has Kähler rank 1.

Proof: We have to show that M cannot admit a Kähler metric. By absurd, if M admitted a Kähler form Ω , then, as ω_0 is exact, $\omega_0 \wedge \Omega = -d(\theta^c \wedge \Omega)$ was an exact volume form, which is impossible by Stokes' theorem. Hence M is non-Kähler. ■

Recall that a **Hopf surface** is a finite quotient of H , where H is a quotient of $\mathbb{C}^2 \setminus 0$ by a polynomial contraction. A Hopf surface is **diagonal** if this polynomial contraction is expressed by a diagonal matrix.

Compact surfaces of Kähler rank 1 have been classified in [CT] and [Br]. They can be:

1. Non-Kähler elliptic fibrations,
2. Diagonal Hopf surfaces and their blow-ups,
3. Inoue surfaces and their blow-ups.

Inoue surfaces have $b_1 = 1$ and hence they are of LCK rank 1. Thus, if they admit an automorphic potential, this has to be proper. But all compact LCK manifolds with proper potential can be deformed to Vaisman ones, [OV2], and Inoue surfaces are not diffeomorphic to Vaisman manifolds (this result follows from a classification of Vaisman surfaces by F. Belgun, [Be]). Thus Inoue surfaces cannot have an automorphic potential.

A cover of a blow-up of any complex manifold cannot admit plurisubharmonic functions because, by the lifting criterion, the projective spaces contained in the blow-up lift to the cover. Thus blow-ups cannot have global potential.

We are left with non-Kähler elliptic fibrations and diagonal Hopf surfaces which are known to admit Vaisman metrics, see *e.g.* [Be]. And hence:

Proposition 3.3: All compact LCK surfaces with potential and with semi-positive form ω_0 admit Vaisman metrics. ■

For further use it is convenient to list all criteria used to distinguish Vaisman Hopf surfaces from non-Vaisman ones.

Theorem 3.4: Let M be a Hopf surface. Then the following are equivalent.

- (i) M is Vaisman.
- (ii) M is diagonalizable.
- (iii) M has Kähler rank 1.
- (iv) M contains at least two distinct elliptic curves.

Proof: Equivalence of the first three conditions is proven above. The equivalence of (iv) and (ii) is shown by Iku Nakamura and Masahide Kato ([N, Theorem 5.2]). Note that the cited result refers to primary Hopf surfaces, but we can always pass to a finite covering and the number of elliptic curves will not change because the eigenvectors for rationally independent eigenvalues cannot be mutually exchanged, and if they were dependent, they would produce infinitely many elliptic curves. ■

3.2 Algebraic groups and the Jordan-Chevalley decomposition

In this section we fix an n -dimensional complex vector space V .

Lemma 3.5: Let $A \in GL(V)$ be a linear operator, and $\langle A \rangle$ the group generated by A . Denote by G the Zariski closure of $\langle A \rangle$ in $GL(V)$. Then, for any $v \in V$, the Zariski closure Z_v of the orbit $\langle A \rangle \cdot v$ is equal to the usual closure of $G \cdot v$.

Proof: Clearly, Z_v is G -invariant. Indeed, its normalizer $N(Z_v)$ in $GL(V)$ is an algebraic group containing $\langle A \rangle$, hence $N(Z_v)$ contains G . The converse is also true: since $\langle A \rangle$ normalizes $\langle A \rangle \cdot v$, its Zariski closure G normalizes the Zariski closure Z_v of the orbit. Therefore, the orbit $G \cdot v$ is contained in Z_v . Since $G \cdot v$ is a constructible set, its Zariski closure coincides with its usual closure, [H], [Kol]. This gives $\overline{G \cdot v} \subset Z_v$. As $\overline{G \cdot v}$ is algebraic and contains $\langle A \rangle \cdot v$, the inclusion $Z_v \subset \overline{G \cdot v}$ is also true. ■

Let $G \subset GL(V)$ be an algebraic group over \mathbb{C} . Recall that an element $g \in G$ is called **semisimple** if it is diagonalizable, and **unipotent** if $g = e^n$, where n is a nilpotent element of its Lie algebra.

Theorem 3.6: (Jordan-Chevalley decomposition, [H, Section 15]) For any algebraic group $G \subset GL(V)$, any $g \in G$ can be represented as a product of two commuting elements $g = g_s g_u$, where g_s is semisimple, and g_u unipotent. Moreover, this decomposition is unique and functorial under homomorphisms of algebraic groups.

Corollary 3.7: Let M be a submanifold of a linear Hopf manifold $H = (V \setminus 0)/A$, $\tilde{M} \subset V \setminus 0$ its \mathbb{Z} -covering, and G the Zariski closure of $\langle A \rangle$ in $GL(V)$. Then \tilde{M} contains the G -orbit of each point $v \in \tilde{M}$. Moreover, G is a product of $G_s := (\mathbb{C}^*)^k$ and a unipotent group G_u commuting with G_s , and both of these groups preserve \tilde{M} .

Proof: Let X be the closure of \tilde{M} in \mathbb{C}^N . The ideal I_X of X is generated by polynomials, as shown in [OV1, Proof of Theorem 3.3]. As the polynomial ring is Noetherian, I_X is finitely generated, [AM]. Therefore, X is a cone of a projective variety.

Consider the smallest algebraic group G containing A . Then G acts naturally on X and preserves it. The last assertion of Corollary 3.7 is implied by the Jordan-Chevalley decomposition. ■

3.3 Finding surfaces in LCK manifolds with potential

Lemma 3.8: Let M be a submanifold of a linear Hopf manifold $H = (V \setminus 0)/A$, $\dim_{\mathbb{C}} M \geq 3$, and $G = G_s G_u$ the Zariski closure of $\langle A \rangle$ with its Jordan-Chevalley decomposition. Then M contains a surface M_0 , with G_u acting non-trivially on its \mathbb{Z} -covering $\tilde{M}_0 \subset V$.

Proof: Another form of this statement is proven by Masahide Kato ([Kat]).

We shall use induction on dimension of M . To prove Lemma 3.8 it would suffice to find a subvariety $M_1 \subset M$ of codimension 1 such that G_u acts non-trivially on its \mathbb{Z} -covering $\tilde{M}_1 \subset \mathbb{C}^n \setminus 0$. Replacing V by the smallest A -invariant subspace containing \tilde{M} , we may assume that the intersection $\tilde{M} \cap V_1 \neq V_1$ for each proper subspace $V_1 \subset V$. Now take a codimension 1 subspace $V_1 \subsetneq V$ which is A -invariant and such that G_u acts on V_1 non-trivially (equivalently, such that A acts on V_1 non-diagonally). Using the Jordan decomposition of A , such V_1 is easy to construct. Then $\tilde{M}'_1 := V_1 \cap \tilde{M}$ gives a subvariety of M of codimension 1 and with non-trivial action of G_u . ■

The same argument gives the following corollary, also parallel to a theorem by Ma. Kato.

Corollary 3.9: Let M be an LCK manifold with potential. Then M has a flag of embedded subvarieties $M \supset M_1 \supset M_2 \supset \cdots \supset M_{\dim M - 1}$ with $\text{codim } M_i = i$. ■

Remark 3.10: This corollary can also be used to show that the Oeljeklaus-Toma manifolds cannot admit LCK structure with potential. Indeed, Oeljeklaus-Toma manifolds do not admit complex curves ([Ver]). For the definition of Oeljeklaus-Toma manifolds, please see [OT].

Lemma 3.11: Let $M \subset H = (V \setminus 0)/\langle A \rangle$ be a surface in a Hopf manifold, and $G = G_s G_u$ the Zariski closure of $\langle A \rangle$ with its Jordan-Chevalley decomposition. Assume that G_u acts on the \mathbb{Z} -covering \tilde{M} non-trivially. Then M is non-diagonal.

Proof: Replacing G by its quotient by the subgroup acting trivially on \tilde{M} if necessary, we may assume that G acts properly on a general orbit in \tilde{M} . Then G is at most 2-dimensional. However, it cannot be 1-dimensional because G_s contains contractions (hence cannot be 0-dimensional) and G_u acts non-trivially. Therefore, $G_s \simeq \mathbb{C}^*$ and $G_u \simeq \mathbb{C}$.

Since G_s acts by contractions, the quotient $S := \tilde{M}/G_s$ is a compact curve, equipped with G_u -action which has a dense orbit. The group G_u can act non-

trivially only on a genus 0 curve, and there is a unique open orbit O of G_u , with $S \setminus O$ being one point. All complex subvarieties of M are by construction G -invariant, and the complement of an open orbit is an elliptic curve, hence M has only one elliptic curve, and is non-diagonalizable by Theorem 3.4. ■

Theorem 3.12: A non-Vaisman compact LCK manifold M with potential contains an embedded non-diagonal Hopf surface.

Proof: Let M be a compact LCK manifold with potential, $\dim_{\mathbb{C}} M \geq 3$. Then M is holomorphically embedded into a Hopf manifold $\mathbb{C}^N \setminus 0 / \langle A \rangle$, where $A \in GL(N, \mathbb{C})$ is a linear operator, see [OV1, Theorem 3.4]. Applying Lemma 3.8 and Lemma 3.11, we find a non-diagonal Hopf surface in M . ■

As an application, we now prove the following characterization of Vaisman manifolds:

Corollary 3.13: Let (M, I) be a compact LCK manifold with potential. Assume the Hermitian form ω_0 is semi-positive definite. Then (M, I) is of Vaisman type.

Proof: If $\dim_{\mathbb{C}} M = 2$, this is just Proposition 3.3.

If $\dim_{\mathbb{C}} M \geq 3$, M contains an embedded non-diagonal Hopf surface H^2 . Then ω_0 restricts to a semi-positive definite (1,1) form on H^2 . By Proposition 3.3, H^2 is Vaisman, and hence diagonal, contradiction. ■

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LIVIU ORNEA

UNIVERSITY OF BUCHAREST, FACULTY OF MATHEMATICS,

14 ACADEMIEI STR., 70109 BUCHAREST, ROMANIA, *and*

INSTITUTE OF MATHEMATICS “SIMION STOILOW” OF THE ROMANIAN ACADEMY,

21, CALEA GRIVITEI STREET 010702-BUCHAREST, ROMANIA

liviu.ornea@imar.ro, lornea@fmi.unibuc.ro

MISHA VERBITSKY

LABORATORY OF ALGEBRAIC GEOMETRY, FACULTY OF MATHEMATICS,

NATIONAL RESEARCH UNIVERSITY HIGHER SCHOOL OF ECONOMICS, 7 VAVILOVA STR. MOSCOW, RUSSIA, *and*

UNIVERSITÉ LIBRE DE BRUXELLES, DÉPARTEMENT DE MATHÉMATIQUE

CAMPUS DE LA PLAINE, C.P. 218/01, BOULEVARD DU TRIOMPHE

B-1050 BRUSSELS, BELGIUM

verbit@mccme.ru